Journal of Algebra **226,** 977–989 (2000)
doi:10.1006/jabr.1999.8213, available online at http://www.idealibrary.com on IDEAL®

Irreducible Modules for Classical and Alternating Groups

William J. Husen

Communicated by Jan Saxl

Received October 6, 1999

1. INTRODUCTION

In this article we consider the following question: Given groups $H < G$ and an absolutely irreducible $\mathbb{F}G$-module M, when does $M{\downarrow_H}$ remain irreducible? This question arises naturally in the study of the maximal subgroups of the classical groups. We assume that $H \cong A_n$ or $2.A_n$, $n \geq 15$, and that G is a simply connected classical algebraic group over $\mathbb{F}$, where $\mathbb{F}$ is an algebraically closed field of characteristic $p \geq 0$. In particular, $G = SL(V)$, $Sp(V)$, or $Spin(V)$, where V is a vector space over $\mathbb{F}$. For convenience, we assume that $\dim(V)$ is even whenever $G = Spin(V)$ and $p = 2$. In this context we prove the following theorem:

THEOREM 1.1. *Suppose that $M = M(\lambda)$ is a tensor indecomposable $\mathbb{F}G$-module such that $M{\downarrow_H}$ is absolutely irreducible and* $\dim(M) > \dim(V)$.

1. *If $G = SL(V)$ then either*

    (a) $\dim(V) \leq n^3$ *or*

    (b) λ *or* $\rho(\lambda) = 3\lambda_1,\ 2\lambda_1 + \lambda_2,\ 2\lambda_1 + \lambda_\ell,\ \lambda_2 + \lambda_\ell$, *or* λ_3 *and* $\dim(V) \leq n^4$.

2. *If $G = Sp(V)$ or $Spin(V)$ then either*

    (a) $\dim(V) \leq 2n^4$ *or*

    (b) $\lambda = 3\lambda_1,\ 2\lambda_1 + \lambda_2$, *or* λ_3 *and* $\dim(V) \leq 2n^6$.

Furthermore, if $n \geq 74$ and $p \neq 2$, then $G \neq Sp(V)$.

0021-8693/00 $35.00

This result strengthens the main result of [3]. Throughout we will assume that λ is the highest weight of M as an $\mathbb{F}G$-module and that M satisfies the hypothesis of Theorem 1.1. As M is tensor indecomposable, we may replace M with a Galois conjugate and assume that λ is a p-restricted dominant weight when $p > 0$. Then we proceed as follows: Using a result of Premet, suitably extended, we find a vector of weight μ in M which has a small stabilizer W_μ in the Weyl group W of G. Then $[W : W_\mu] \leq \dim(M)$. Conversely, we produce a large subgroup $\widetilde{C}$ of H such that $M{\downarrow_{\widetilde{C}}}$ contains a small submodule M_0. Using Frobenius reciprocity, we get $\dim(M) \leq \dim(M_0)[H : \widetilde{C}]$. The result follows from comparing these bounds.

Our result is analogous to that achieved in [5] for H a classical group. In similar notation, they find an upper bound for $\dim(M)$ in terms of $\dim(V)$. This is in contrast to our result, where we find an upper bound for $\dim(V)$ in terms of n. This reflects the fact that there is an infinite family of examples where $\dim(V) < n$ and $M{\downarrow_H}$ is irreducible. In particular, if V is the minimal $n-1$ or $n-2$ dimensional module for H over $\mathbb{F}$, $G = Spin(V)$ or $SL(V)$, and $M = \wedge^k V$, $k < \frac{n}{2}$, then $M{\downarrow_H}$ is absolutely irreducible.

2. NOTATION AND PRELIMINARY RESULTS

We use the following notation throughout this article:

ℓ is the Lie rank of G and $e = \lfloor \log_n(\ell) \rfloor$.

$\overline{e} = e$ or $\frac{3e}{2}$ if $e \leq 3$ or $e > 3$, respectively.

T is a fixed maximal torus of G.

If $G = Spin(V)$ or $Sp(V)$, we write $t = (t_1, \ldots, t_\ell)$ for an arbitrary $t \in T$.

If $G = SL(V)$, we write $t = (t_1, \ldots, t_{\ell+1})$ for $t \in T$, considering T as a subgroup of $GL(V)$.

$\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ are the simple roots of G with respect to T, ordered as in [2].

Φ is the root system generated by Π and Φ^+ is the set of positive roots.

$\{\lambda_1, \ldots, \lambda_\ell\}$ are the fundamental dominant weights of G ordered according to the ordering of Π. For convenience, we define $\lambda_0 = \lambda_{\ell+1} = 0$.

If $G = SL(V)$ then ρ is the permutation of Π and $\{\lambda_1, \ldots, \lambda_\ell\}$ induced by the non-trivial symmetry of the Dynkin diagram of G.

M_μ is the μ-weight space of M.

Λ is the weight lattice of G and Λ^+ is the set of dominant weights in Λ.

$\lambda = \sum_{i=1}^{\ell} a_i \lambda_i$ is the highest weight of M.

$k_s = \sum_{i<\ell/2} i a_i$, $k_d = \sum_{i \geq \ell/2} (\ell - i + 1) a_i$, and $k_d^* = \ell + 1 - k_d$.

W is the Weyl group of G and W_μ is the stabilizer of $\mu \in \Lambda$ in W.

ϵ_i is the element of $\mathrm{Hom}(T, F^*)$ such that $\epsilon_i(t) = t_i$.

It will be convenient to know α_i and λ_i in terms of ϵ_i:

$G = SL(V)$:

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \qquad \lambda_i = \sum_{j=1}^{i} \epsilon_j.$$

$G = Sp(V)$:

$$\alpha_i = \begin{cases} \epsilon_i - \epsilon_{i+1}, & i < \ell, \\ 2\epsilon_\ell, & i = \ell, \end{cases} \qquad \lambda_i \sum_{j=1}^{i} \epsilon_j.$$

$G = Spin(V)$, $\dim(V)$ odd:

$$\alpha_i = \begin{cases} \epsilon_i - \epsilon_{i+1}, & i < \ell, \\ \epsilon_\ell, & i = \ell, \end{cases} \qquad \lambda_i = \begin{cases} \displaystyle\sum_{j=1}^{i} \epsilon_j, & i < \ell, \\ \displaystyle\frac{1}{2}\sum_{j=1}^{\ell} \epsilon_j, & i = \ell. \end{cases}$$

$G = Spin(V)$, $\dim(V)$ even:

$$\alpha_i = \begin{cases} \epsilon_i - \epsilon_{i+1}, & i < \ell, \\ \epsilon_{\ell-1} + \epsilon_\ell, & i = \ell, \end{cases} \qquad \lambda_i = \begin{cases} \displaystyle\sum_{j=1}^{i} \epsilon_j, & i < \ell - 1, \\ \displaystyle\frac{1}{2}\sum_{j=1}^{\ell} \epsilon_j, & i = \ell - 1, \\ \displaystyle\frac{1}{2}\left(\sum_{j=1}^{\ell-1} \epsilon_j - \epsilon_\ell\right), & i = \ell. \end{cases}$$

Note that, by considering $T < GL(V)$, we have a simpler characterization of λ_i in terms of the coordinate functions ϵ_i when $G = SL(V)$.

For $\mu \in \Lambda^+$, write $\mu = \sum_{i=1}^{\ell} b_i \lambda_i$ and define $m_s(\mu) = \max\{0, i \mid l < \frac{\ell}{2} \text{ and } b_i \neq 0\}$, $m_d(\mu) = \max\{0, \ell - i + 1 \mid i \geq \frac{\ell}{2} \text{ and } b_i \neq 0\}$, and $m_d^*(\mu) = \ell + 1 - m_d(\mu)$.

If U is an $\mathbb{F}$-vector space, we write $U^{\otimes i} = \otimes_{j=1}^{i} U$ and define $U^{\otimes 0} = \mathbb{F}$.

$\{v_i\}$ is a basis for V such that $t \in T$ acts on this basis as $\mathrm{diag}(t_1, \ldots, t_{\ell+1})$ if $G = SL(V)$, as $\mathrm{diag}(t_1, \ldots, t_\ell, t_\ell^{-1}, \ldots, t_1^{-1})$ if $G = Sp(V)$ or $Spin(V)$ with $\dim(V)$ even, and as $\mathrm{diag}(t_1, \ldots, t_\ell, 1, t_\ell^{-1}, \ldots, t_1^{-1})$ if $G = Spin(V)$ with $\dim(V)$ odd.

If $G = SL(V)$, then $\{v_i^*\}$ is a basis for V^* such that $tv_i^* = t_i^{-1} v_i^*$. Note that $v_{\ell+1}^*$ is a vector of weight λ_ℓ.

Our initial result concerns binomial coefficients:

LEMMA 2.1. *Suppose that* i_1, i_2, *and* j *are integers.*

1. *If* $1 \le i_1 \le j$ *then*
$$\binom{j}{i_1} \ge \left(\frac{j-i_1+1}{i_1}\right)^{i_1}.$$
2. *If* $e \ge 3$, $j \ge n^e$, *and* $\frac{n}{e} \le i_1 \le \frac{j}{2}$ *then* $\binom{j}{i_1} > \dim(M)$.
3. *If* $0 \le i_1 \le i_2 \le j$ *then*
$$\frac{j!}{i_1!i_2!(j-i_1-i_2)!} \ge \binom{j}{i_1+i_2}.$$

Proof. Part (1) follows by writing
$$\binom{j}{i_1} = \prod_{m=1}^{i_1} \frac{j-m+1}{i_1-m+1} \ge \prod_{m=1}^{i_1} \frac{j-i_1+1}{i_1} = \left(\frac{j-i_1+1}{i_1}\right)^{i_1}.$$
For (2), $\frac{n}{e} \le i_1 \le \frac{j}{2}$ and $j \ge n^e$ implies that
$$\binom{j}{i_1} \ge \binom{j}{n/\overline{e}} \ge \binom{n^e}{n/\overline{e}}.$$
Applying (1) to the last expression gives
$$\binom{j}{i_1} \ge \left(\frac{n^e - n/\overline{e} + 1}{n/\overline{e}}\right)^{n/\overline{e}}.$$
As $e \ge 3$, we have
$$n^e - \frac{n}{\overline{e}} + 1 > \frac{n^e}{\overline{e}} \qquad \text{and} \qquad \frac{e-1}{\overline{e}} \ge \frac{1}{2}.$$
This gives
$$\binom{j}{i_1} > \left(\frac{n^e/\overline{e}}{n/\overline{e}}\right)^{n/\overline{e}} \ge n^{n/2}.$$
Now, M is an irreducible $\mathbb{F}H$-module, so $\dim(M)$ is bounded above by $\chi(1)$ for some $\chi \in \mathrm{Irr}(H)$. Using the orthogonality of such characters, $\chi(1) < \sqrt{|H|} \le \sqrt{n!} < n^{n/2}$, hence $\dim(M) < n^{n/2}$. This proves (2). Part (3) follows by expanding
$$\binom{j}{i_1+i_2} = \frac{j!}{(i_1+i_2)!(j-i_1-i_2)!}$$
and comparing denominators. In particular, $(i_1+i_2)! \ge i_1!i_2!$, hence the result. ∎

LEMMA 2.2. *Suppose that* $e \ge 3$ *and* $\mu \in \Lambda^+$. *If* M *possesses a vector of weight* μ, *then* $m_s(\mu) + m_d(\mu) < \frac{n}{e}$. *Furthermore,* $m_d(\mu) = 0$ *if* $G = Sp(V)$ *or* $Spin(V)$.

Proof. Write $\mu = \sum_{i=1}^{\ell} b_i\lambda_i$. If $b_i > 0$ then $W_\mu \leq W_{\lambda_i}$, which implies that $\dim(M) \geq [W : W_\mu] \geq [W : W_{\lambda_i}]$. Let $G = Sp(V)$ or $Spin(V)$ so that $[W : W_{\lambda_i}] = 2^i\binom{\ell}{i}, 2^{\ell-3}(\ell^2 - \ell)$, or $2^{\ell-1}$, the latter two holding if and only if $G = Spin(V)$ with $\dim(V)$ even and $i = \ell - 2$ or $i > \ell - 2$, respectively. If $0 \neq m_d(\mu) \leq \ell - 3$ then $b_{m_d^*(\mu)} > 0$ and $\dim(M) \geq 2^{m_d^*(\mu)}\binom{\ell}{m_d^*(\mu)} \geq 2^{m_d^*(\mu)} \geq 2^{\ell/2} \geq 2^{n^3/2} > \dim(M)$, which is a contradiction. Similar calculations handle the case $m_d(\mu) \geq \ell - 2$. Therefore $m_d(\mu) = 0$ and $m_s \neq 0$, so that $\dim(M) \geq 2^{m_s(\mu)}\binom{\ell}{m_s(d)} \geq \binom{\ell}{m_s(\mu)}$. Since $m_s(\mu) \leq \frac{\ell}{2}$, part (2) of Lemma 2.1 implies that $\binom{\ell}{m_s(\mu)} > \dim(M)$ if $m_s(\mu) \geq \frac{n}{e}$, hence $m_s(\mu) < \frac{n}{e}$.

Now suppose that $G = SL(V)$, so that $[W : W_{\lambda_i}] = \binom{\ell+1}{i}$. Since $m_d^*(\mu) = \ell + 1 - m_d(\mu)$, we have $\binom{\ell+1}{m_d^*(\mu)} = \binom{\ell+1}{m_d(\mu)}$ and $m_d(\mu) \leq \frac{\ell}{2}$. Arguing as before, $\binom{\ell+1}{m_d(\mu)} > \dim(M)$ if $m_d(\mu) \geq \frac{n}{e}$. Thus $m_d(\mu) < \frac{n}{e}$. Similarly, $m_s(\mu) < \frac{n}{e}$. Thus, as $e \geq 3$, $m_s(\mu) + m_d(\mu) < \frac{\ell}{2}$. Now $W_\mu < W_{\lambda_{m_s(\mu)}} \cap W_{\lambda_{m_d^*(\mu)}}$ so that

$$\begin{aligned}[W : W_\mu] &\geq [W : W_{\lambda_{m_s(\mu)}} \cap W_{\lambda_{m_d^*(\mu)}}] \\ &= \frac{(\ell+1)!}{m_s(\mu)!m_d(\mu)![\ell + 1 - m_s(\mu) - m_d(\mu)]!}.\end{aligned}$$

Using part (3) of Lemma 2.1, we get $[W : W_\mu] \leq \binom{\ell+1}{m_s(\mu)+m_d(\mu)}$. Again, Lemma 2.1 forces $m_s(\mu) + m_d(\mu) \leq \frac{n}{e}$. ∎

Note that this lemma implies that $k_d = 0$ when $G = Sp(V)$ or $Spin(V)$. The last lemma in this section is a technical one:

LEMMA 2.3. *Suppose that* $k_t \leq \frac{n}{e}$ *and that* $r = 5,\ 7,\ 11,$ *or* 13.

1. *If* $e \geq 3$ *then* $r\lceil k_t/(r-1)\rceil < n - 2$.
2. *If* $e \geq 4$ *then* $r\lceil 2k_t/(r-1)\rceil < n - 2$.

Proof. We will give a proof for the case $e \geq 3$ and $r = 7$, all other cases following from an identical argument. As $\overline{e} \geq e$, we may assume that $k_t \leq \frac{n}{3}$. By way of contradiction, assume that $r\lceil k_t/(r-1)\rceil \geq n - 2$. As $r = 7$, $7\lceil k_t/(r-1)\rceil < 7\,(k_t/6 + 1)$. Then $n - 2 < 7k_t/6 + 7 \leq 7n/18 + 7$. Thus $n < 144/11 < 14$, which is a contradiction. ∎

3. WEIGHT MODULES

Let $V_{\mathbb{Z}}$ be the $\mathbb{Z}$-lattice generated by $\{v_i\}$ and let $\overline{V} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. Moreover, assume that $\overline{G}$ is the complex version of G so that $\overline{G} = SL(\overline{V}),\ Sp(\overline{V})$, or $Spin(\overline{V})$. Let $\mathscr{L}$ be the adjoint module for $\overline{G}$, which is a complex Lie algebra, and let $\mathscr{U}$ be the universal enveloping algebra of $\mathscr{L}$. We will assume that $\{\lambda_i\}$ are the fundamental dominant weights for $\mathcal{U}$, that Π is the corresponding set of simple roots, and that Φ is the root system of $\mathcal{U}$.

Let $\{x_\alpha,\ \alpha \in \Phi;\ h_j,\ 1 \leq j \leq \ell\}$ be a Chevalley basis for $\mathcal{L}$. The Kostant $\mathbb{Z}$-form $\mathcal{U}_{\mathbb{Z}}$ is generated by $\{x_\alpha^m/m! \mid \alpha \in \Phi,\ m \in \mathbb{Z}^+\}$. Moreover, the hyperalgebra $\mathcal{H}$ corresponding to G is defined to be $\mathcal{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}$.

Define $v_{\lambda_i} = v_1 \wedge \cdots \wedge v_i$, which is naturally an element of $\wedge^i V_{\mathbb{Z}}$ as well as of $\wedge^i V$. For convenience, we identify these two elements. Set $\overline{v}_{\lambda_i} = v_{\lambda_i} \otimes_{\mathbb{Z}} 1_{\mathbb{C}}$. Then $\overline{v}_{\lambda_i}$ is a vector of weight λ_i unless $G = Spin(V)$ and $i \geq \ell - 1$.

LEMMA 3.1. *$\wedge^i V$ and $\wedge^{\dim(V)-i} V^*$ contain a highest weight vector of weight λ_i unless $G = Spin(V)$ and $i \geq \ell - 1$.*

Proof. By [1, Sects. 15.3, 17.3, and 19.5], $\left(\wedge^i \overline{V}\right) \cap \mathcal{K}$ is the $\mathbb{F}\mathcal{U}$-module with highest weight λ_i, where $\mathcal{K}$ is the intersection of kernels of contraction maps. Moreover, as $\overline{v}_{\lambda_i} \in \mathcal{K}$, it is a highest weight vector. If $\mathcal{U}_{\mathbb{Z}}$ is the Kostant $\mathbb{Z}$-form of $\mathcal{U}$, it follows that $\mathcal{U}_{\mathbb{Z}} v_{\lambda_i} \otimes_{\mathbb{Z}} \mathbb{F}$ is the Weyl module for G of highest weight λ_i. Since $v_{\lambda_i} \in \wedge^i V$, $\wedge^i V$ contains a highest weight vector of weight λ_i, and we have the first claim. The second follows from the $\mathbb{F}SL(V)$-isomorphism $\wedge^i V \cong \wedge^{\dim(V)-i} V^*$. ∎

As a direct result of this lemma we have:

PROPOSITION 3.2. *M occurs as a section of*

1. *$V^{\otimes k_s} \otimes V^{*\otimes k_d}$ if $G = SL(V)$ or*
2. *$V^{\otimes k_s}$ if $G = Sp(V)$ or $Spin(V)$.*

Proof. First note that if v_{μ_1} and v_{μ_2} are highest weight vectors for G of weights μ_1 and μ_2, respectively, then $v_{\mu_1} \otimes v_{\mu_2}$ is a highest weight vector of weight $\mu_1 + \mu_2$. In particular, if $\mathcal{H}^+$ are the elements of $\mathcal{H}$ corresponding to positive root elements and $x \in \mathcal{H}^+$, then $x(v_{\mu_1} \otimes v_{\mu_2}) = (xv_{\mu_1}) \otimes v_{\mu_2} + v_{\mu_1} \otimes (xv_{\mu_2}) = 0$. From the universal property of Weyl modules, it is enough to show that the modules above contain a highest weight vector of weight λ. Let $I = \{i | a_i \neq 0\}$ and write $M_i = \wedge^i V$ and $M_i' = \wedge^i V^*$. Consider the module $\otimes_I M_i^{a_i}$. From the above, the highest weight of this module is $\sum a_i \lambda_i = \lambda$. The claim follows by recalling the definition of k_s and k_d, and replacing $M_{\ell-i+1}$ with M_i' when $G = SL(V)$ and $i \geq \ell/2$. ∎

The next lemma extends the results of Premet [6] and Suprunenko [7].

LEMMA 3.3. *Let $\mu \neq 0$ be a subdominant weight of λ. If $G = Sp(V)$ and $p = 2$, assume further that $\mu = \sum b_i \lambda i$ with $b_\ell = 0$ and that α_ℓ is not in the support of $\lambda - \mu$. Then M possesses a vector of weight μ.*

Proof. By Premet's result [6], we need only consider the case where $p = 2$ and $G = Sp(V)$. Define $\Pi_0 = \{\alpha_1, \ldots, \alpha_{\ell-1}\}$ and let Φ_0 be the root system generated by Π_0. Suppose $\mathcal{U}_0$ is the subalgebra of $\mathcal{U}$ corresponding to Π_0; then $(\mathcal{U}_0)_{\mathbb{Z}}$ is generated by $\{x_\alpha^m/m! \mid \alpha \in \Phi_0,\ m \in \mathbb{Z}^+\}$. Note that $\mathcal{U}_0$ is the universal enveloping algebra of a complex Lie algebra of type $A_{\ell-1}$. Write $\Phi^+ = \{\beta_1, \ldots, \beta_j\}$ such that $\beta_i \in \Phi_0^+$ if $i \leq |\Phi_0^+|$.

Now, M is a homomorphic image of $\mathcal{U}_{\mathbb{Z}} v_\lambda \otimes_{\mathbb{Z}} \mathbb{F}$ where v_λ is a highest weight vector of weight λ. Moreover, M_μ is the image of the space $\langle (\prod x_{-\beta_i}^{m_i}/m_i!) v_\lambda \mid \lambda - \mu = \sum m_i \beta_i \rangle$. From our restrictions on λ and μ, it is clear that $m_i = 0$ if $\beta_i \notin \Phi_0$. In particular, M_μ is a weight space contained in the image of $(\mathcal{U}_0)_{\mathbb{Z}} v_\lambda \otimes_{\mathbb{Z}} \mathbb{F}$. By [7], this space is non-trivial; hence M contains a vector of weight μ. ∎

LEMMA 3.4. *M possesses a vector of weight $\lambda_{k_s} + \lambda_{k_d^*}$.*

Proof. Recall that $k_d = 0$ when $G = Sp(V)$ or $Spin(V)$ and that $\lambda_0 = \lambda_{\ell+1} = 0$. By the previous lemma, it is enough to show that $\lambda_{k_s} + \lambda_{k_d^*}$ is a subdominant weight of λ and that $\lambda - \lambda_{k_s} - \lambda_{k_d^*}$ is supported on $\{\alpha_1, \ldots, \alpha_{\ell-1}\}$ when $G = Sp(V)$ and $p = 2$. Using the descriptions from Section 2 write $\lambda = b_i \epsilon_i$ and note that $b_1 \geq b_2 \geq \cdots \geq b_m$ where $m = \max\{i | b_i \neq 0\}$. We will consider the case $G = Sp(V)$ or $Spin(V)$ first. Here $\sum b_i = k_s$ so that $k_s - m \geq 0$. We will induct on $k_s - m$. If $k_s - m = 0$ then $m = k_s$ and $\lambda = \lambda_{k_s}$, so the claim follows in this case. Assume $k_s - m > 0$. This implies that $b_i > 1$ for some i, so pick m_0 to be maximal subject to $b_{m_0} > 1$. By Lemma 2.2, $m < k_s < \frac{n}{e} < \ell - 1$ so that $\alpha = \epsilon_{m_0} - \epsilon_{m+1}$ is a positive root. Moreover, using the descriptions from Section 2, $\lambda - \alpha = \sum b_i \epsilon_i - \epsilon_{m_0} + \epsilon_{m+1} = \mu$ is a dominant weight. As $m + 1 < \ell$, $\lambda - \mu$ is supported on $\{\alpha_1, \ldots, \alpha_{\ell-1}\}$. By induction, λ_{k_s} is a subdominant weight of μ, hence of λ. Moreover, $\lambda - \lambda_{k_s}$ contains no α_ℓ term. Thus the result holds in this case.

Now suppose that $G = SL(V)$. Here we write $\lambda = \mu_1 + \mu_2$, where $\mu_1 = \sum_{i<\ell/2} b_i \lambda_i$ and $\mu_2 = \sum_{i \geq \ell/2} b_i \lambda_i$. The previous argument shows that λ_{k_s} is a subdominant weight of μ_1 and that λ_{k_d} is a subdominant weight of $\rho(\mu_2)$. Thus $\rho(\lambda_{k_d}) = \lambda_{k_d^*}$ is a subdominant weight of μ_2 and $\lambda_{k_s} + \lambda_{k_d^*}$ is a subdominant weight of $\mu_1 + \mu_2 = \lambda$. ∎

Note that Lemma 2.2 implies that $k_s + k_d \leq \frac{n}{e}$.

4. THE SUBGROUP $\widetilde{C}$ AND SUBMODULE M_0

Assume that $r \neq p$ is an odd prime and that $k > 0$ is an integer such that $kr < n - 2$. Let H_1 be a subgroup of $H/Z(H)$ such that $H_1 \cong S_{n-2}$ and let $E_{r,k}$ be a subgroup of H_1 generated by k disjoint r-cycles. Then $E_{r,k}$ is an elementary abelian r-group of rank k. Let $C = C_{H_1}(E_{r,k})$ and $N = N_{H_1}(E_{r,k})$. Then $C = E_{r,k} C_1$ where $C_1 \cong S_{n-kr-2}$ and $N = N_1 C_1$ where $N_1 \cong \mathbb{Z}_r : \mathbb{Z}_{r-1} \wr S_k$ and $[N_1, C_1] = 1$. The action of N on $E_{r,k}$ is easy to see; in particular, if $1 \neq \sigma \in E_{r,k}$ then $\sigma^N = \sigma^{N_1}$ and $\sigma^i \in \sigma^N$ for $(i, r) = 1$. Moreover, if σ is the product of m disjoint r-cycles, then $C_N(\sigma) \cong \mathbb{Z}_r \wr S_m \times \mathbb{Z}_r : \mathbb{Z}_{r-1} \wr S_{k-m} \times C_1$. Thus $|\sigma^N| = (r-1)^m\binom{k}{m}$. As $\binom{k}{m} \geq k$ for $0 < m < k$ and $(r-1)^{k-1} \geq k$ for all r, k as above, we have that $|\sigma^N| = (r-1)^m\binom{k}{m} \geq (r-1)k$.

Let $\widetilde{E}_{r,k}$, $\widetilde{C}$, $\widetilde{C}_1$, $\widetilde{N}$, and $\widetilde{N}_1$ be the full preimage in H of $E_{r,k}$, C, C_1, N, and N_1, respectively. Since $r > 2$, we may identify $E_{r,k} = O_r(\widetilde{E}_{r,k})$ and note that the action of $\widetilde{N}$ on $E_{r,k}$ is the same as the action of N on $E_{r,k}$, so the above observations still hold. Define $E^*_{r,k} = \mathrm{Hom}(E_{r,k}, \mathbb{F}^*)$. Then the $\widetilde{N}$ action on this subgroup is just the dual of its action on $E_{r,k}$. If $1 \neq \varphi \in E^*_{r,k}$, we abuse notation and write φ^i for the element $\varphi^i\colon\ \sigma \mapsto \varphi(\sigma^i)$ of $E^*_{r,k}$.

Now, $V\!\downarrow_{E_{r,k}} = \bigoplus_{\varphi \in E^*_{r,k}} V_\varphi$, where V_φ is the homogeneous component of φ on V. Moreover, each V_φ is an $\mathbb{F}\widetilde{C}$-module.

LEMMA 4.1. *Assume that* $V_{\varphi_1}, V_{\varphi_2} \neq 0$. *Then*

1. *If* $\varphi_1 \in \varphi_2^{\widetilde{N}}$ *then* $V_{\varphi_1} \cong V_{\varphi_2}$ *as* $\mathbb{F}\widetilde{C}_1$*-modules.*
2. *If* $\varphi_1 \neq 1$ *then* V_{φ_1} *is a self-dual* $\mathbb{F}\widetilde{C}_1$*-module.*
3. $V_{\varphi_1} \perp V_{\varphi_2}$ *if* $\varphi_1 \neq \varphi_2^{-1}$.
4. *If* $\varphi_i \neq 1$ *then* V_{φ_i} *is totally singular.*

Proof. From the earlier comments, $\varphi_2^{\widetilde{N}} = \varphi_2^{\widetilde{N}_1}$ and $[\widetilde{C}_1, \widetilde{N}_1] = 1$. This implies that $V_{\varphi_1} \cong V_{\varphi_2}$ as $\mathbb{F}\widetilde{C}_1$-modules. Let $\sigma \in \widetilde{N}_1$ be an element which interchanges φ_1 and φ_1^{-1}. As $\left(V_{\varphi_1}\right)^* \cong V_{\varphi_1^{-1}}$ as an $\mathbb{F}E_{r,k}$-module, $V_{\varphi_1} \oplus \left(V_{\varphi_1}\right)^*$ is a module for $\left(E_{r,k}\langle\sigma\rangle\right)\widetilde{C}_1$. As σ centralizes $\widetilde{C}_1$, $V_{\varphi_1} \cong V^*_{\varphi_1}$ as $\mathbb{F}\widetilde{C}_1$-modules, proving (2). We may assume that $G = Sp(V)$ or $Spin(V)$ for the remaining parts. Let $\mathbf{f}$ be the non-degenerate, G-invariant, bilinear form on V and let Q be the associated quadratic form when $G = Spin(V)$. Suppose $u_i \in V_{\varphi_i}$ so that $gu_i = \varphi_i(g)$ for all $g \in E_{r,k}$. Then $\mathbf{f}(u_1, u_2) = \mathbf{f}(gu_1, gu_2) = \varphi_1(g)\varphi_2(g)\mathbf{f}(u_1, u_2)$ for all $g \in E_{r,k}$. If $\mathbf{f}(u_1, u_2) \neq 0$ then $\varphi_1 = \varphi_2^{-1}$, hence (3) is true. (4) follows from (3) by taking $\varphi_1 = \varphi_2$, unless $G = Spin(V)$ and $p = 2$. In this case, however, $Q(u_1) = Q(gu_1) = \varphi_1^2(g)Q(u_1)$. As $r > 2$, $\varphi_1^2 = 1$ if and only if $\varphi_1 = 1$. Thus $Q(u_1) = 0$, proving (4). ∎

From this point forward, we will assume that $r = 5,\ 7$, or 11 and $(r,\ p) = 1$. Also, we assume that $e \geq 3$ if $G = SL(V)$ and that $e \geq 4$ if $G = Sp(V)$ or $Spin(V)$. Let $k_t = k_s + k_d$. By the remark following Lemma 3.4, $k_t < \frac{n}{e}$. By Lemma 2.3, if $G = SL(V)$, then $r\lceil k_t/(r-1)\rceil < n-2$. Similarly, if $G = Sp(V)$ or $Spin(V)$, then $r\lceil 2k_t/(r-1)\rceil < n-2$. Therefore we can take $k = \lceil k_t/(r-1)\rceil$ if $G = SL(V)$ and $k = \lceil 2k_t/(r-1)\rceil$ if $G = Sp(V)$ or $Spin(V)$.

Let $1 \neq \varphi \in E^*_{r,k}$ such that $V_\varphi \neq 0$. Set $\varphi_1 = \varphi$ and write $\varphi^{\widetilde{N}} = \{\varphi_1, \ldots, \varphi_m\}$ such that $\varphi_i = \varphi^{-1}_{m-i+1}$, so that $\oplus_{i=1}^{m/2} V_{\varphi_i}$ is a totally singular subspace. Recall from the initial remarks in this section that $m \geq (r-1)k$. Hence $m \geq k_t$ when $G = SL(V)$ and $m/2 \geq k_t$ when $G = Sp(V)$ or $Spin(V)$.

By replacing T with a suitable conjugate, we may assume that each V_{φ_i} with $i \leq m/2$ has a basis which is a subset of $\{v_j\}$ where $j \leq \dim(V)/2$. Moreover, when $G = SL(V)$, we may assume that $E_{r,k} < T$. Write $V^*\downarrow_{E_{r,k}} = \oplus_{\psi \in E^*_{r,k}} V'_\psi$. Because $t \in T$ acts as t^{-1} on V^* and $E_{r,k} < T$, we have that $V'_\psi \cong V_{\psi^{-1}}$ as $\mathbb{F}E_{r,k}$-modules. In particular, if $\{v_{i_1}, \ldots, v_{i_j}\}$ is a basis for V_ψ then $\{v^*_{i_1}, \ldots, v^*_{i_j}\}$ is a basis for V'_ψ. Moreover, $V'_\psi \cong V_\psi$ as $\mathbb{F}\widetilde{C}_1$-modules as V_ψ is a self-dual $\mathbb{F}\widetilde{C}_1$-module.

Define $U_i = V_{\varphi_i}$ if $i \leq k_s$ and $U_i = V'_{\varphi_i}$ if $k_s < i \leq k_s + k_d$.

LEMMA 4.2. *$M\downarrow_{\widetilde{C}}$ possesses a submodule isomorphic to $U = \bigotimes_{i=1}^{k_t} U_i$.*

Proof. From Proposition 3.2, M occurs as a section of $V^{\otimes k_s} \otimes V^{*\otimes k_d}$. Thus it contains an $\mathbb{F}G$-submodule $\overline{M}$ such that M is a homomorphic image of $\overline{M}$. Let ω be the corresponding homomorphism of $\overline{M}$. Recall that $k_d = 0$ when $G = Sp(V)$ or $Spin(V)$.

Notice that $\otimes_{i \leq k_s} U_i \subset V^{\otimes k_s}$ and $\otimes_{i > k_s} U_i \subset V^{*\otimes k_d}$, so $U \subset V^{\otimes k_s} \otimes V^{\otimes k_d}$. Each basic tensor in $V^{\otimes k_s} \oplus V^{*\otimes k_d}$ is a weight vector for T. Moreover, U has a basis consisting of basic tensors. Since the tensor factors in U are distinct, every basic tensor in U corresponds to a distinct weight. Pick $u_i \in U_i$ such that $u_i \in \{v_j\}$ or $u_i \in \{v^*_j\}$ and let $u = \otimes_{i=1}^{k_t} u_i$. It is clear that u_i is a weight vector. Because of our definition of U_i, in particular because the v_i and v^*_j that occur all have distinct indices, there is an element $w \in W$ such that $u_i = v_i$ if $i \leq k_s$ and $u_i = v^*_{\ell+1-i}$ if $k_s < i \leq k_s + k_d$. This means that u^w is a vector of weight $\lambda_{k_s} + \lambda_{k^*_d}$. Thus every element of U is conjugate, under W, to u^w.

Now recall from [3, Section 2] that the group $\mathcal{S} = \mathcal{S}_{k_s} \times \mathcal{S}_{k_d}$ acts on $V^{\otimes k_s} \otimes V^{*\otimes k_d}$ by permuting the tensor factors. A straightforward calculation, which we omit, shows that $\langle \mathbb{F}\mathcal{S}u^w \rangle$ is the entire $\lambda_{k_s} + \lambda_{k^*_d}$ weight space of $V^{\otimes k_s} \otimes V^{*\otimes k_d}$. Using Lemma 3.4, we see that $\omega\big(\langle \mathbb{F}\mathcal{S}u^w \rangle \cap \overline{M}\big)$ must be a non-trivial subspace of M. Pick $\sigma \in \mathbb{F}\mathcal{S}$ such that $\sigma u^w \in \overline{M}$ and $\omega(\sigma u^w) \neq 0$. We claim that $\sigma U \subset \overline{M}$ and that $\omega|_{\sigma U}$ is injective.

First, $\overline{M}$ is an $\mathbb{F}G$-module containing σu^w. Thus, if w' is any element of W, then $(\sigma u^w)^{w'} = \sigma u^{ww'}$. This is because the action of G commutes with the action of $\mathcal{S}$. Since $u \in U$ was arbitrary and all basic vectors of U are W-conjugate to u, $\sigma U \subset \overline{M}$. Now suppose that $u' \neq u$ is any basic tensor in U. Then u' and u are vectors of distinct weights, and so are σu and $\sigma u'$. In particular, σU has a basis consisting of vectors of distinct weights.

Moreover, their images under ω remain distinct and span a subspace of M of dimension $\dim(U)$. Thus $\omega|_{\sigma U}$ is injective. Note that this implies that σU is isomorphic to U as an $\mathbb{F}\widetilde{C}$-module. As σU is an $\mathbb{F}\widetilde{C}$-module and ω is a homomorphism, we have that $\omega(\sigma U)$ is an $\mathbb{F}\widetilde{C}$-submodule of M which is isomorphic to σU, hence to U. ∎

LEMMA 4.3. *Let $\delta = 0$ if k_t is even and let $\delta = 1$ if k_t is odd. Then M possesses an $\mathbb{F}\widetilde{C}$-submodule M_0 such that* $\dim(M_0) \leq \ell^{\delta}$.

Proof. Note that $k_t \geq 2$ so that $\dim(U_i) \leq \ell$. From Lemma 4.1, V_{φ_i} is self-dual as an $\mathbb{F}\widetilde{C}_1$-module. Similarly, V'_{φ_i} is also self-dual. Moreover, $V_{\varphi_i} \cong V'_{\varphi_i}$ as $\mathbb{F}\widetilde{C}_1$-modules. We induct on $|k_t|$. If $k_t = 1$ then the result is immediate. If $|k_t| = 2$ then $U = U_1 \otimes U_2$, where U_1 and U_2 are equivalent, self-dual $\mathbb{F}\widetilde{C}_1$-modules. Hence $\widetilde{C}_1$ stabilizes a one-dimensional subspace $\overline{M}_0$ of U. If $k_t > 2$ then write $U = U_1 \otimes U_2 \otimes U'$. As U' has $k_t - 2$ tensor factors, by induction $U'\downarrow_{\widetilde{C}_1}$ has a subspace $\overline{M}_1$ of dimension at most ℓ^{δ}. As $E_{r,k}$ acts as scalars on U, $\widetilde{C}$ stabilizes $\overline{M}_0 \otimes \overline{M}_1$. Take M_0 to be the image of $\overline{M}_0 \otimes \overline{M}_1$ in M under the homomorphism from Lemma 4.2. ∎

5. PROOF OF THEOREM 1.1

In this section we begin the proof of Theorem 1.1. Using Lemma 4.3 and Frobenius reciprocity, we have $\dim(M) \leq \ell^{\delta}[H : \widetilde{C}]$. In particular, $\dim(M) \leq \ell^{\delta}(n!/r^k(n-rk-2)!)$. And, by Lemma 3.4, $[W : W_{\lambda_{k_s}+\lambda_{k_d^*}}] \leq \dim(M)$. By combining these inequalities, we have

$$[W : W_{\lambda_{k_s}+\lambda_{k_d^*}}] \leq \ell^{\delta}\frac{n!}{r^k(n-rk-2)!}. \tag{1}$$

Throughout this section we will assume that $r = 7$ if $p = 5$ and $r = 5$ otherwise.

Case 1. $G = SL(V)$ and $k_t \geq 13$. Using part (3) of Lemma 2.1,

$$[W : W_{\lambda_{k_s}+\lambda_{k_d^*}}] \geq \binom{\ell+1}{k_s+k_d} = \binom{\ell+1}{k_t}.$$

Using part (1) of Lemma 2.1,

$$\binom{\ell+1}{k_t} \geq \left(\frac{\ell-k_t+2}{k_t}\right)^{k_t}.$$

As in the previous case, $\ell - k_t + 2 < \frac{\ell}{2}$. Then using inequality (1), we get

$$\left(\frac{\ell}{2k_t}\right)^{k_t} < \ell^{\delta}\frac{n!}{r^k(n-rk-2)!}. \tag{2}$$

So

$$\ell^{k_t-\delta} < \frac{n!}{r^k(n-rk-2)}(2k_t)^{k_t} < n^{rk+2}(2k_t)^{k_t}.$$

As before $\ell \geq n^3$ so that $1 < n^{rk+2-3k_t+3\delta}(2k_t)^{k_t}$. When $r = 5$,

$$rk \leq 5\frac{k_t+3}{4} \qquad \text{and} \qquad rk+2-3k_t+3\delta \leq \frac{35-7k_t}{4}.$$

When $r = 7$,

$$rk \leq 7\frac{k_t+5}{6} \qquad \text{and} \qquad rk+2-3k_t+3\delta \leq \frac{65-11k_t}{6}.$$

In both cases $rk + 2 - 3k_t + 3\delta < -k_t$ when $k_t \geq 14$. As $k_t < \frac{n}{e}$, $n^{rk+2-3k_t+3\delta}k_t^{k_t} < 1$ if $k_t \geq 13$, hence the result in this case.

Case 2. $G = SL(V)$ and $k_t \leq 12$. For $k_t \geq 6$, we explicitly compute k and δ for $r = 5,\ 7$, and 11. We then evaluate inequality (2) for these values. In all cases, we have a contradiction for at least two primes. For $k_t = 3$ or $k_t = 4$, we assume $\ell \geq n^4$. As $8 < n$, we may use $r = 3$ as well as $r = 5$. Again using inequality (2) with these primes, we get a contradiction. For $k_t = 5$, $11 < n$ so we may use $r = 3$ as well as $r = 7$. Here we use inequality (1), in particular,

$$\binom{\ell+1}{5} > \frac{\ell^\delta}{9}\frac{n!}{(n-11)!}.$$

Finally, the results of [4] eliminate the possibility $k_t = 2$. This completes part (1) of Theorem 1.1.

Case 3. $G = Sp(V)$ or $Spin(V)$ and $k_t \geq 16$. In this case, $k_t = k_s$ so that $[W : W_{\lambda_{k_s}+\lambda_{k_d^*}}] = 2^{k_t}\binom{\ell}{k_t}$. By part (1) of Lemma 2.1,

$$2^{k_t}\binom{\ell}{k_t} \geq 2^{k_t}\left(\frac{\ell-k_t+1}{k_t}\right)^{k_t}.$$

Since $\ell \geq n^4$ and $k_t < \frac{n}{e}$, $\ell - k_t + 1 > \frac{\ell}{2}$. Thus

$$2^{k_t}\binom{\ell}{k_t} > \left(\frac{\ell}{k_t}\right)^{k_t}.$$

Using inequality (1), we get

$$\left(\frac{\ell}{k_t}\right)^{k_t} < \ell^\delta\frac{n!}{r^k(n-rk-2)!}. \tag{3}$$

Then

$$\ell^{k_t-\delta} < \frac{n!}{r^k(n-rk-2)} k_t^{k_t} < n^{rk+2} k_t^{k_t}.$$

As $\ell \geq n^4$, we have $n^{4k_t-4\delta} < n^{rk+2} k_t^{k_t}$ and $1 < n^{rk+2-4k_t+4\delta} k_t^{k_t}$. When $r = 5$,

$$rk \leq 5\frac{k_t+1}{2} \qquad \text{and} \qquad rk + 2 - 4k_t + 4\delta \leq \frac{13-3k_t}{2}.$$

When $r = 7$,

$$rk \leq 7\frac{k_t+2}{3} \qquad \text{and} \qquad rk + 2 - 4k_t + 4\delta \leq \frac{32-5k_t}{3}.$$

In both cases $rk + 2 - 4k_t + 4\delta < -k_t$ when $k_t \geq 16$. As $k_t < \frac{n}{e}$, $n^{rk+2-4k_t+4\delta} k_t^{k_t} < 1$ if $k_t \geq 16$, hence the result in this case.

Case 4. $G = Sp(V)$ or $Spin(V)$ and $1 < k_t \leq 15$. When $k_t = 4, 6$ or $k_t \geq 8$, we explicitly compute k and δ for $r = 5, 7, 11$, and 13. We then evaluate inequality (3) for these values. In all cases we end up with a contradiction for at least two distinct values of r. We omit these calculations as they are straightforward. The remaining possibilities for k_t, namely, $k_t = 2, 3, 5$, and 7, require subtler arguments. When $k_t = 2$, it is clear that $8 < n$. Here we may take $r = 3$ or $r = 5$. These choices for r lead to a contradiction as before. When $k_t = 5$, we take $r = 5$ and this gives a contradiction, except when $p = 5$. For this case we take $r = 11$ and work with inequality (1). In particular, we have $2^5\binom{\ell}{5} < \ell\frac{n!}{11(n-13)!}$. As $\ell \geq n^4$, this inequality leads to a contradiction. When $k_t = 7$, $n \geq 42$. Again $19, 21 < n$ so we may take $r = 17$ or $r = 19$ in this case. Here $k = 1$ and we have a contradiction. Finally, if $k_t = 3$ then $n \geq 18$. If $\ell > n^6$ then we get a contradiction using $r = 3$ and $r = 7$. This completes part (2) of Theorem 1.1.

Finally, suppose that $G = Sp(V)$, $p > 2$, and that V is an $\mathbb{F}A_n$-module. As no irreducible $\mathbb{F}A_n$-module has an invariant symplectic form, V must be is reducible. Let V_0 be an irreducible submodule. Then, as before, there can be no invariant symplectic form on V_0, hence it is totally singular. Thus H is contained in a parabolic subgroup of G, hence of $GL(M)$. As this is not the case, $V{\downarrow_H}$ must have one constituent which is a faithful $\mathbb{F}2.A_n$-module. By [8], $\dim(V)$ is divisible by $2^{(n-s-1)/2}$ where s is the number of 1's in the binary expansion of n. In particular, $\dim(V) > 2^{(n-3)/2}/\sqrt{n}$. By parts (1) and (2) of Theorem 1.1, we have that $\dim(V) \leq 2n^6$. Comparing these bounds we see that $n < 74$. This completes the proof of Theorem 1.1. ∎

REFERENCES

1. W. Fulton and J. Harris, "Representation Theory," Springer-Verlag, Berlin/New York, 1991.
2. J. E. Humphreys, "Introduction to Lie Algebras and Representation Theory," Springer-Verlag, Berlin/New York, 1972.
3. W. J. Husen, Restrictions of $\Omega_m(q)$-modules to alternating groups, *Pacific J. Math.*, to appear.
4. K. Magaard and G. Malle, Irreducibility of alternating and symmetric squares, *Manuscripta Math.* **95** (1998), 169–180.
5. K. Magaard, G. Röhrle, and D. Testerman, On maximal subgroups of finite classical groups, preprint.
6. A. Premet, Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic, *Mat. Sb.* **61** (1988), 167–183 [English translation].
7. I. Suprunenko, The invariance of the weight system of irreducible representations of algebraic groups and Lie algebras of type A_l with restricted highest weights, under reduction modulo p, *Vestsī Akad. Navuk Belarusī Ser. Fīz. Mat. Navuk* **2** (1983), 18–22 [in Russian].
8. A. Wagner, An observation on the degrees of projective representations of symmetric and alternating groups over an arbitrary field, *Arch. Math.* **29** (1977), 583–589.